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Kaehlerian Manifolds on Einstein-Recurrent curvature tensor

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Abstract: Bochner (1949) obtained some results on curvature and betti numbers. Hasegawa (1974) studied on H projective recurrent Kaehlerian manifolds and Bochner recurrent Kaehlerian manifolds. Also, Negi (2016) obtained Some Recurrence Properties in Kaehlerian, Einstein Kaehlerian and Tachibana Spaces. After that, Negi et. al. (2019) studied on Projective recurrent and symmetric tensor in almost Kaehlerian spaces. Again, Negi and Sulochana (2021) have studied on conformal symmetric tensor of kaehlerian manifolds. In this paper, the author calculated some properties of Kaehlerian Manifolds on Einstein-Recurrent curvature tensor and computes the relations between special type curvature tensors.
Key words: Kaehlerian manifolds, H-conharmonic recurrent tensor, Riemannian manifolds, Ricci-recurrent spaces.
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Introduction:

Let the real dimension Kaehlerian manifold K^n $(n > 3)$ and Kaehlerian structure is denoted by (g, F), where g is Riemannian metric and F is complex structure, we define:

$$F_i^{\ a} F_a^{\ j} = -\delta_i^{\ j}, \quad F_{ji} = -F_{ih} \ (F_{ji} = F_j^{\ a} g_{ai}), \ F_{i,a}^{\ \ j} = 0, \tag{1.1}$$

where (,) denoted the covariant differentiation with respect to the metric tensor g_{ij} and $F_i^{\ j}$ is mixed tensor.

The Riemannian curvature tensor will be denoted by R^h_{kji} , then:

$$R^h_{kji} = \delta_k \begin{Bmatrix} h \\ ji \end{Bmatrix} - \delta_j \begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ ka \end{Bmatrix}\begin{Bmatrix} a \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ ja \end{Bmatrix} \tag{1.2}$$

Similarly, the Ricci tensor and ricci scalar curvature are define in this form:

$R_{ji} = R^a_{aji}$ and $R = g^{ji} R_{ji}$.

If we define a tensor S_{ji} by $S_{ji} = F_j^{\ a} R_{ai}$, Then, we have $S_{ji} = -S_{ij}$.

Holomorphically projective curvature tensor which parallel to Weyl's projective curvature tensor and Bochner (Conformal) curvature tensor in terms of real coordinates has been defined by Tachibana (1967) as follows:

$$P_{kjih} = R_{kjih} + \frac{1}{n+2}[R_{ki}g_{jh} - R_{ji}g_{kh} + S_{ki}F_{jh} - S_{ji}F_{kh} + 2S_{kj}F_{ih}]. \tag{1.3}$$

$$B_{kjih} = R_{kjih} + \frac{1}{n+4}[R_{ki}g_{jh} - R_{ji}g_{kh} + g_{ki}R_{jh} - g_{ji}R_{kh}$$
$$+ S_{ki}F_{jh} - S_{ji}F_{kh} + F_{ki}S_{jh} - F_{ji}S_{kh} + 2S_{kj}F_{ih} + 2F_{kj}S_{ih}]$$

$$-\frac{R}{(n+2)(n+4)}[g_{ki}g_{jh}-g_{ji}g_{kh}+F_{ki}F_{jh}-F_{ji}F_{kh}+2F_{kj}F_{ih}]. \quad (1.4)$$

Again, for $K^n(n>3)$ a tensor U_{kjih} name as the H-concircular curvature tensor given by Tachibana (1967).

$$U_{kjih}=R_{kjih}+\frac{R}{n(n+2)}[g_{ki}g_{jh}-g_{ji}g_{kh}+F_{ki}F_{jh}-F_{ji}F_{kh}+2F_{kj}F_{ih}], \quad (1.5)$$

If the tensor U_{kji}^{h} vanishes identically, then this is the constant holomorphic sectional curvature Kaehlerian manifold. The H-projective curvature tensor and the Bochner curvature tensor coincide together with the H- concircular curvature tensor iff K^n is an Einstein-recurrent curvature manifolds.

2. Kaehlerian Manifolds on Einstein-Recurrent curvature tensor.

Definition 2.1 The Kaehlerian manifold $K^n(n>3)$ is called H-conharmonic curvature Kaehlerian manifolds if:

$$T_{kjih,l}-\lambda_l T_{kjih}=0, \quad \text{where} \quad \lambda_l\neq 0 \quad (2.1)$$

The Einstein curvature tensor of K^n which is homogeneous to the Ricci tensor, then

$$E_{ji}=R_{ji}=\frac{R}{n}g_{ji}. \quad (2.2)$$

And the Einstein-recurrent curvature Kaehlerian manifold described by:

$$E_{ij,l}-\lambda_l E_{ij}=0, \quad (2.3)$$

Now, the Ricci-recurrent curvature, Bochner recurrent curvature, H-projective recurrent curvature and H-concircular recurrent curvature and H-conharmonic recurrent curvature Kaehlerian manifold are respectively following identities holds:

$$R_{ji,l}-\lambda_l R_{ji}=0, \quad (2.4)$$

$$B_{kjih,l}-\lambda_l B_{kjih}=0, \quad (2.5)$$

$$P_{kjih,l}-\lambda_l P_{kjih}=0, \quad (2.6)$$

$$U_{kjih,l}-\lambda_l U_{kjih}=0. \quad (2.7)$$

From (2.4), the Ricci-recurrent Kaehlerian manifold satisfies:

$$R_{,l}-\lambda_l R=0. \quad (2.8)$$

Every Kaehlerian Ricci-recurrent curvature manifold is Einstein-recurrent curvature tensor but converse is not necessarily true.

A Kaehlerian Einstein-recurrent curvature manifold strictly:

$$R_{ji,l}-\lambda_l R_{ji}=\frac{1}{n}(R_{,l}-\lambda_1 R)\,g_{ji}.$$

Then, using of (2.8) we have,

$$R_{ji,l}-\lambda_l R_{ji}=0. \quad (2.9)$$

Now, we have the following:

Theorem 2.1 A Kaehlerian Einstein-recurrent curvature manifold to be Kaehlerian Ricci-recurrent with the identical reappearance vector then conformal curvature tensor has the following forms:

$$B^h_{kji} = R^h_{kji} + \frac{1}{n+4}[L_{ki}\delta^h_j - L_{ji}\delta^h_k + g_{ki}L^h_j - g_{ji}L^h_k + M_{ki}F^h_j - M_{ji}F^h_k + F_{ki}M^h_j - F_{ji}M^h_k + 2M_{kj}F^h_i + 2F_{kj}M^h_i] \qquad (2.10)$$

Proof. We have if Ricci- recurrent Kaehlerian manifold satisfies, then by using (2.8), we get:

$$R_{,l} - \lambda_l R = 0$$

Assume that, the conharmonic curvature tensor is:

$$L_{ji} = R_{ji} - \frac{R}{2(n+2)}g_{ji} \qquad (2.11)$$

and the Ricci curvature tensor is:

$$M_{ji} = F_j{}^a L_{ai} = S_{ji} - \frac{R}{2(n+2)}F_{ji}\,, \qquad (2.12)$$

Then, from (2.11) and (2.12) we get the equation (2.10) of Bochner curvature tensor.

Theorem 2.2 Every Kaehlerian Einstein-recurrent curvature manifold satisfies this relation between Bochner recurrent curvature tensor and the H-concircular recurrent curvature tensor is:

$$B_{kjih,l} - \lambda_l B_{kjih} = U_{kjih.l} - \lambda_l U_{kjih}. \qquad (2.13)$$

Proof. We know that the Einstein-recurrent manifold, we have

$$R_{ji,l} - \lambda_l R_{ji} = \frac{1}{n}(R_{,l} - \lambda_l \mathrm{R})g_{ji}. \qquad (2.14)$$

Then by (2.11) and (2.14), we have

$$L_{ji,l} - \lambda_l L_{ji} = \frac{n+4}{2n(n+2)}(R_{,l} - \lambda_l \mathrm{R})g_{ji}. \qquad (2.15)$$

Consequently (2.15) with $F_k{}^i$, then we obtain

$$M_{ji.l} - \lambda_l M_{ji} = \frac{n+4}{2n(n+2)}(R_{,l} - \lambda_l \mathrm{R})F_{ji}. \qquad (2.16)$$

Now, by (2.12), (2.14) and (2.15), then

$$B_{kjih,l} - \lambda_l B_{kjih} = R_{kjih,l} - \lambda_l R_{kjih} + \frac{R_{,l} - \lambda_l R}{n(n+2)}[g_{jh}g_{ki} - g_{kh}g_{ji} + F_{jh}F_{ki} - F_{kh}F_{ji} + 2F_{kj}F_{ih}], \qquad (2.17)$$

By (1.5), we get:

$$B_{kjih,l} - \lambda_l B_{kjih} = U_{kjih.l} - \lambda_l U_{kjih}.$$

Hence theorem 2.2 proved.

Theorem 2.3 Every Kaehlerian Einstein-recurrent curvature manifold of constant holomorphic sectional curvature tensor is Bochner recurrent curvature tensor with the same indices.

Proof. We know that, the Kaehlerian manifold is H-concircular recurrent, then

$$R_{kjih,l} - \lambda_l R_{kjih} + \frac{R_{,l} - \lambda_l R}{n(n+2)}[g_{ki}g_{jh} - g_{ji}g_{kh} + F_{ki}F_{jh} - F_{ji}F_{kh} + 2F_{kj}F_{ih}] = 0 \quad (2.18)$$

Consequently (2.13) with g^{ki}, we find:

$$R_{jh,l} - \lambda_l R_{jh} = \frac{R_{,l} - \lambda_l R}{n} g_{jh}, \quad (2.19)$$

from (2.2), we get:

$$E_{ji,l} = \lambda_l E_{ji}. \quad (2.20)$$

Therefore, we furthermore calculated every H-concircular recurrent Kaehlerian manifold is Einstein-recurrent curvature Kaehlerian manifolds; Kaehlerian manifold (K^n) of H-projective recurrent curvature is the H-concircular recurrent curvature; every H-projective recurrent curvature Kaehlerian manifold is Einstein-recurrent curvature Kaehlerian manifold with same indices but the converse is not necessarily true.

Theorem 2.4 Every Kaehlerian Einstein-recurrent manifold designate H-projective recurrent Kaehlerian manifold is that the Bochner recurrent curvature manifold with the identical vector of reappearance.

Proof. In view of theorem (2.2) and (2.3), we can prove the above theorem (2.4).

Theorem 2.5 The conditions for H-concircular recurrent curvature Kaehlerian manifold and H-projective recurrent curvature Kaehlerian manifold are that comparable to the Bochner recurrent curvature Kaehlerian manifold is:

$P_{kjih,l} - \lambda_l P_{kjih} = U_{kjih,l} - \lambda_l U_{kjih}$ and $B_{kjih,l} - \lambda_l B_{kjih} = P_{kjih,l} - \lambda_l P_{kjih}$.

Proof. We know that, the Einstein-recurrent Kaehler manifold is:

$$S_{ji,l} - \lambda_l S_{ji} = \frac{R_{,l} - \lambda_l R}{n} F_{ji}. \quad (2.21)$$

By (1.3), then we get:

$$P_{kjih,l} - \lambda_1 P_{kjih} = R_{kjih,l} - \lambda_l R_{kjih}$$
$$+ \frac{1}{n+2}[(R_{ki,l} - \lambda_1 R_{ki})g_{jh} - (R_{ji,l} - \lambda_l R_{ji})g_{kh}$$
$$+ (S_{ki,l} - \lambda_l S_{ki})F_{jh} - (S_{ji,l} - \lambda_l S_{ji})F_{kh} + 2(S_{kj,l} - \lambda_l S_{kj})F_{ih}]. \quad (2.22)$$

Using (2.20) and (2.21) in (2.22), we get

$$P_{kjih,l} - \lambda_l P_{kjih} = R_{kjih,l} - \lambda_l R_{kjih}$$
$$+ \frac{R_{,l} - \lambda_l R}{n(n+2)}[g_{ki}g_{jh} - g_{ji}g_{kh} + F_{ki}F_{jh} - F_{ji}F_{kh} + 2F_{kj}F_{ih}], \quad (2.23)$$

Therefore, we get:

$$P_{kjih,l} - \lambda_l P_{kjih} = U_{kjih,l} - \lambda_l U_{kjih}. \quad (2.24)$$

Hence, we proved the result.

Similarly, we can prove the result by using theorem 2.2 and 2.5, then

$$B_{kjih,l} - \lambda_l B_{kjih} = P_{kjih,l} - \lambda_l P_{kjih}. \quad (2.25)$$

Now, we proved the following theorems systematically:

Theorem 2.6 If a Kaehlerian manifold (K^n), satisfied any two of the following properties:

1. The space is Bochner recurrent curvature tensor,
2. The space is H-Concircular recurrent curvature tensor,
3. The space is Einstein-recurrent curvature tensor.

Then it must be also satisfying the third.

Proof. Taking equation (1.4) and (1.5) with using equation (2.5) and (2.7), after that simplify by (2.3), (2.4) and (2.9), then we get the result.

Theorem 2.7 If a Kaehlerian manifold (K^n), satisfied any two the following properties:

1. The space is Bochner recurrent curvature tensor,
2. The space is H-Conharmonic recurrent curvature tensor,
3. The space is Einstein-recurrent curvature tensor.

Then it must be also satisfying the third.

Theorem 2.8 If a Kaehlerian manifold (K^n), satisfied any two the following properties:

1. The space is H-Concircular recurrent curvature tensor,
2. The space is H-Conharmonic recurrent curvature tensor,
3. The space is Einstein-recurrent curvature tensor.

Then it must be also satisfying the third.

Theorem 2.9 If a Kaehlerian manifold (K^n), satisfied any two the following properties:

1. The space is H-Projective recurrent curvature tensor,
2. The space is Bochner recurrent curvature tensor,
3. The space is Einstein-recurrent curvature tensor.

Then it must be also satisfying the third.

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